T.D.C. II

PART- III (3)

Taylor's Theorem for Functions of Two variables :-

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Prestudy :- At first, we require to study of limit, continuity and partial derivatives of several variables (specially functions of two variables). Study of Taylor's theorem of function of one variable is also useful for basic concept and present study.

Statement :- If $f(x,y)$ is a function which possesses continuous partial derivatives of order n in any domain of point (a,b) and the domain its large enough to contain a point $(a+h, b+k)$ within it, then $\exists$ a +ve number, $0<\theta<1$, so that

$$f(a+h,b+k) = f(a,b) + \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b) + \frac{1}{\lfloor 2}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)$$
$$+ \cdots + \frac{1}{\lfloor n-1}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a,b) + R_n$$

where $R_n = \frac{1}{\lfloor n}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k)$; $0<\theta<1$.

Proof :- Let $x = a+th$, $y = b+tk$; where $0 \le t \le 1$ is a parameter and $f(x,y) = f(a+th, b+tk) = \phi(t)$ —— (i)

Since the partial derivatives of $f(x,y)$ of order n are continuous in the domain under consideration, $\phi^n(t)$ is continuous in $[0,1]$ and also

$$\phi'(t) = \frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$
$$= h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$
$$\phi'(t) = \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f$$

As $x = a+th$; $y = b+tk$

$\frac{dx}{dt} = h$; $\frac{dy}{dt} = k$

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$$\phi''(t) = \frac{d^2f}{dt^2} = \frac{d}{dt}\left(\frac{df}{dt}\right) = \frac{d}{dt}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)$$

$$= h\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) + k\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)$$

$$= h\left\{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{dy}{dt}\right\}$$
$$+ k\left\{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{dy}{dt}\right\}$$

$$= h^2\frac{\partial^2 f}{\partial x^2} + hk\frac{\partial^2 f}{\partial x\partial y} + kh\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}$$

$$\phi''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f$$ (As partial derivatives are continuous)

$\vdots$

$$\phi^n(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f \quad \text{--- (ii)}$$

By Maclaurin's theorem; we know that

$$\phi(t) = \phi(0) + t\,\phi'(0) + \frac{t^2}{\lfloor 2}\phi''(0) + \cdots + \frac{t^{n-1}}{\lfloor n-1}\phi^{n-1}(0)$$
$$+ \frac{t^n}{\lfloor n}\phi^n(\theta t) \quad \text{--- (iii)}$$

where $0 < \theta < 1$

In eqn (iii); using $t=1$; we have

$$\phi(1) = \phi(0) + \phi'(0) + \frac{1}{\lfloor 2}\phi''(0) + \cdots + \frac{1}{\lfloor n-1}\phi^{n-1}(0) + \frac{1}{\lfloor n}\phi^n(\theta) \quad \text{--- (iv)}$$

But $\phi(0) = f(a,b)$

$$\phi'(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(a,b)$$

$\vdots$

$$\phi^{n-1}(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1}f(a,b)$$

--- (V) (using eqn (ii))

and $\phi^n(\theta) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k)$

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With the help of eqn (i) & (v); eqn (iv) gives

$$f(a+h, b+k) = \phi(1) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \cdots$$

$$+ \frac{1}{\lfloor n-1}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a,b) + R_n; \quad \text{—— (vi)}$$

Where $R_n = \frac{1}{\lfloor n}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a+\theta h, b+\theta k); 0<\theta<1$ —— (vii)

R_n is called remainder after n terms & the theorem, Taylor's theorem with remainder or Taylor's expansion about the point (a,b).

<u>Another Form of Taylor's theorem</u>:— The theorem can be stated in still another form:

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) + \cdots$$

$$+ \frac{1}{\lfloor n-1}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^{n-1} f(a,b) + R_n \quad \text{—— (viii)}$$

Where $R_n = \frac{1}{\lfloor n}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^n f[a+(x-a)\theta, b+(y-b)\theta]$, —— (ix)

$0<\theta<1$

called the Taylor's expansion of $f(x,y)$ about the point (a,b) in powers of $(x-a)$ and $(y-b)$.
which is obtained by putting $h=(x-a)$; $k=(y-b)$ in eqns (vi) & (vii).

<u>Example</u>:— (1) Expand $f(x,y) = x^2+xy+y^2$ in powers of $(x-2)$ & $(y-3)$.

<u>Solution</u>:— Here $f(x,y) = x^2+xy+y^2$; —— (1)

$$f(2,3) = 2^2 + 2\times 3 + 3^2 = 4+6+9$$
$$= 19$$

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$$\frac{\partial f}{\partial x} = 2x+y \quad ; \quad \frac{\partial f}{\partial y} = x+2y \; ; \; \frac{\partial^2 f}{\partial x^2} = 2 \; ; \; \frac{\partial^2 f}{\partial y^2} = 2$$

and $\dfrac{\partial^2 f}{\partial x \partial y} = 1$ and the values of third and higher order partial derivatives of f are zero. $\Big\}$ — (ii)

Thus at $(2,3)$;

$$\left(\frac{\partial f}{\partial x}\right)_{(2,3)} = 2\times 2+3 = 4+3 = 7$$

$$\left(\frac{\partial f}{\partial y}\right)_{(2,3)} = 2+2\times 3 = 2+6 = 8$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \; ; \; \frac{\partial^2 f}{\partial x \partial y} = 1 \; ; \; \frac{\partial^2 f}{\partial y^2} = 2$$

— (iii)

and the values of third & higher order partial derivatives of f are zero.

Hence by Taylor's formula

$$f(x,y) = f(2,3) + \left[(x-2)\frac{\partial}{\partial x} + (y-3)\frac{\partial}{\partial y}\right] f(2,3) +$$
$$\frac{1}{\lfloor 2}\left[(x-2)^2 \frac{\partial^2}{\partial x^2} + 2(x-2)(y-3)\frac{\partial^2}{\partial x \partial y} + (y-3)^2 \frac{\partial^2}{\partial y^2}\right] f(2,3)$$
$$+ \cdots \qquad \text{— (iv)}$$

Using values of eqn (iii) in eqn (iv); we have

$$f(x,y) = 19 + [7(x-2) + 8(y-3)] + \frac{1}{2}\left[2(x-2)^2 + 2(x-2)(y-3)\times 1 + 2(y-3)^2\right] + 0$$

$$\Rightarrow f(x,y) = 19 + [7(x-2) + 8(y-3)] + \left[(x-2)^2 + (x-2)(y-3) + (y-3)^2\right]$$

which is required Taylor's expansion.

<u>Exercise</u>:- Expand the function $f(x,y) = x^3 + 3x^2y^2 + 4xy^2 + y^3$ by Taylor's theorem in powers of $(x-1)$ and $(y-1)$.

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